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MULTIPLE SCATTERING OF ANTIPLANE SHEAR WAVES IN A FIBER-REINFORCED COMPOSITE MEDIUM WITH INTERFACIAL LAYERS

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Abstract—This study considers the multiple scattering of antiplane shear waves in a metal matrix composite reinforced by fibers with interfacial layers. We assume same-size cylindrical inclusion and same-thickness interface layers with nonhomogeneous elastic properties. First, the problem of the scattering of plane axial shear waves by a large number N of fibers, arbitrarily distributed in an infinite matrix, is considered. The resulting equations are then averaged, considering the positions of the fibers to be random. The averaged equations are solved by using Lax's quasicrystalline approximation. Numerical calculations for an SiC-fiber-reinforced Al composite are carried out and the effect of interface properties on the phase velocity and attenuation of coherent plane wave, and the effective shear modulus is shown graphically. © 1997 Elsevier Science Ltd

1. INTRODUCTION

Much current practical interest exists concerning wave propagation through a composite medium with a random distribution of inclusions with interface layers (Shindo *et al.*, 1995). The theoretical investigation of the dynamic properties of such composites is a prerequisite to the design of a composite with high strength and high damping. Recently, Shindo and Niwa (1996) analyzed the scattering of antiplane shear waves by a cylindrical inclusion with thick nonhomogeneous interface layer, and applied the results of the single scattering problem to coherent plane wave in a fiber-reinforced metal matrix composite with interface layers. While the scattering of plane elastic waves by a single inclusion has been the subject of much research in the past, the wave propagation problem for composite materials including the effect of multiple scattering by several inclusions has been rather sparsely discussed.

The purpose of this study is to analyze the effects of interface layers and multiple scattering by a distribution of inclusions on the wave propagation of time-harmonic axial shear waves in a fiber-reinforced metal matrix composite. The interface layer is modeled by any number of homogeneous layers, which may be chosen to approximate any nonhomogeneous material properties. The composite medium contains a random distribution of cylindrical inclusions of same size with interface layers of same thickness. The problem of the scattering of plane axial shear waves by a large number N of cylindrical inclusions with interfaces, arbitrarily distributed in an infinite matrix, is analyzed and the resulting equations are then averaged, considering the positions of the inclusions to be random (Bose and Mal, 1973, 1974). The averaged equations are solved by using Lax's quasicrystalline approximation to yield the propagation characteristics of the average wave (Lax, 1952). The particular case when the pair correlation function has an exponential form, is examined in detail. The phase velocity and attenuation of coherent elastic shear wave, and the effective shear modulus are obtained numerically for an SiC fiber-reinforced Al composite and the numerical values are shown in graphs for various interface properties at designated frequencies. The results in the dependence on concentration at wavelength comparable to scatter size here are valid for thick nonhomogeneous interface layers and a wide range of frequencies.

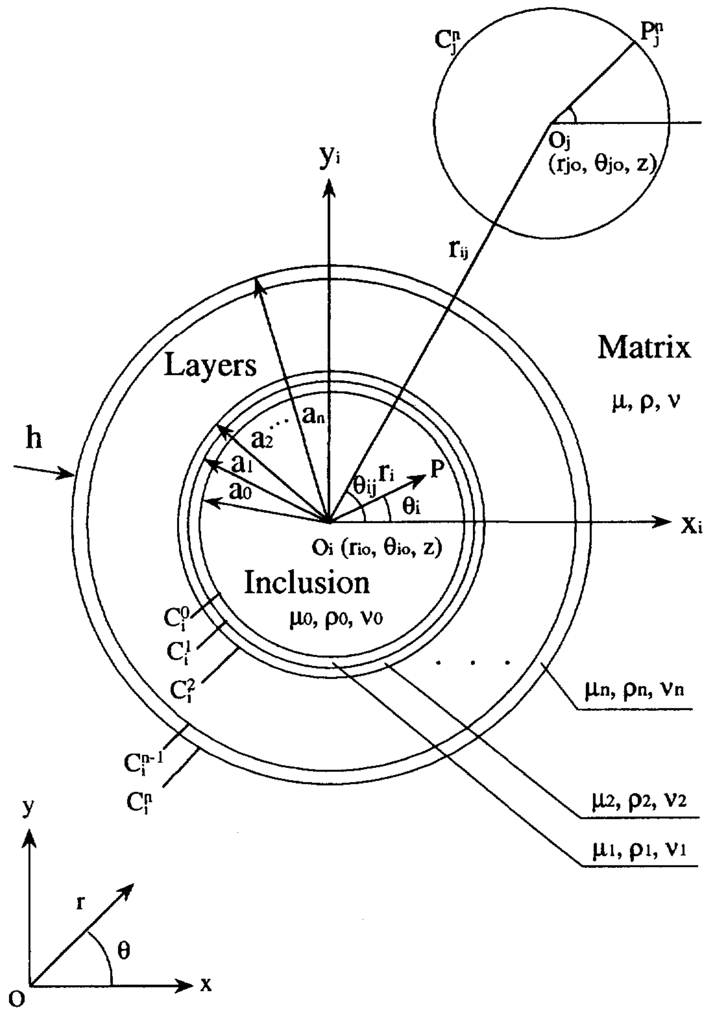

Fig. 1. A cylindrical inclusion with an interface layer and coordinate systems.

2. STATEMENT OF THE PROBLEM AND SCATTERING OF ANTIPLANE SHEAR WAVES BY N INCLUSIONS

We suppose the identical cylindrical inclusions of radius a_0 to be located within a large region S in an infinite matrix. Let μ, ρ, v be the shear modulus, the mass density, the Poisson's ratio of the matrix, and μ_0, ρ_0, v_0 those of the inclusions. We assume that thick layers of uniform thickness h with variable material properties are presented at the interfaces separating the matrix from each cylinder. Let the inclusion be separated from the matrix by n layers. The geometry is depicted in Fig. 1 where (x, y, z) is the Cartesian coordinate system with origin at o and (r, θ, z) is the corresponding cylindrical coordinate system. The layer is subdivided into several thick-walled shells and the material properties within each shell of inner radius a_{l-1}, outer radius $a_l (l = 1 \sim n)$ and uniform thickness $h_l = a_l - a_{l-1}$ are μ_l, ρ_l, v_l. Labelling the inclusions by suffixes $i = 1, 2, \ldots, N$ and taking suitable coordinate axes in a transverse plane, let the boundaries of the ith cylindrical inclusion and the shells be denoted by C_i^l $(l = 0 \sim n)$ and the Cartesian and cylindrical coordinates of those center o_i (r_{io}, θ_{io}, z) be (x_i, y_i, z) and (r_i, θ_i, z), respectively.

The displacement in the z-direction w satisfies the wave equation

$$\nabla^2 w = \frac{1}{c_{sh}^2} \frac{\partial^2 w}{\partial t^2}, \tag{1}$$

where $\nabla^2 = \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + (1/r^2)(\partial^2/\partial\theta^2)$ is the two-dimensional Laplacian operator, t is the time and c_{sh} is the shear wave speed in the matrix,

$$c_{sh} = \left(\frac{\mu}{\rho}\right)^{1/2}. \tag{2}$$

The stress component τ_{rz} is found as

$$\tau_{rz} = \mu\frac{\partial w}{\partial r}. \tag{3}$$

We consider a plane shear (longitudinal shear, SH) wave polarized in the z-direction and propagating in the positive x-direction. Thus,

$$w^i = w_0 \exp\left[i(k_{sh}x - \omega t)\right], \tag{4}$$

where a superscript i stands for the incident component, ω is the circular frequency of the wave and w_0 is the amplitude of the incident SH wave. k_{sh} is the wave number of the SH wave in the matrix,

$$k_{sh} = \frac{\omega}{c_{sh}}. \tag{5}$$

In what follows, the time factor $\exp(-i\omega t)$ will be omitted from all the field quantities.

The displacement fields in the matrix, the lth layer of the ith cylindrical inclusion and the ith cylindrical inclusion may be expressed in the forms

$$w^s = \sum_{i=1}^{N} w_i^s$$

$$w_i^s = \sum_{m=-\infty}^{\infty} A_{im}H_m(k_{sh}r_i)\exp(im\theta_i) \tag{6}$$

$$w_i^l = \sum_{m=-\infty}^{\infty} \left[B_{im}^l H_m(k_{sh}^l r_i)\exp(im\theta_i) + C_{im}^l J_m(k_{sh}^l r_i)\exp(im\theta_i)\right] \quad (l = 1 \sim n) \tag{7}$$

$$w_i^t = \sum_{m=-\infty}^{\infty} D_{im}J_m(k_{sh}^0 r_i)\exp(im\theta_i), \tag{8}$$

where superscripts s, t and l ($l = 1 \sim n$) denote the scattered component within a matrix, the transmitted component within a cylindrical inclusion and the field quantity within an lth layer. A_{im}, B_{im}^l, C_{im}^l and D_{im} are the unknowns to be solved, $H_m()$ is the mth order Hankel function of the first kind and $J_m()$ is the mth order Bessel function of the first kind (Watson, 1966). The wave numbers k_{sh}^l ($l = 1 \sim n$) in the lth layer and k_{sh}^0 in the cylindrical inclusion are given by

$$k_{sh}^l = \frac{\omega}{c_{sh}^l} \quad (l = 1 \sim n)$$

$$k_{sh}^0 = \frac{\omega}{c_{sh}^0}, \tag{9}$$

where the longitudinal shear wave speeds c_{sh}^{l} in the lth layer and c_{sh}^{0} in the cylindrical inclusion are

$$c_{sh}^{l} = \left(\frac{\mu_l}{\rho_l}\right)^{1/2} \quad (l = 1 \sim n)$$

$$c_{sh}^{0} = \left(\frac{\mu_0}{\rho_0}\right)^{1/2}. \tag{10}$$

The boundary conditions on C_j^l $(l = 0 \sim n)$ are

$$w_j^n = w^s + w^i, \quad \tau_{rzj}^n = \tau_{rz}^s + \tau_{rz}^i \quad (r_j = a_n) \tag{11}$$

$$w_j^l = w_j^{l+1}, \quad \tau_{rzj}^l = \tau_{rzj}^{l+1} \quad (r_j = a_l, l = 1 \sim n-1) \tag{12}$$

$$w_j^t = w_j^1, \quad \tau_{rzj}^t = \tau_{rzj}^1 \quad (r_j = a_0), \tag{13}$$

where $\tau_{rzj}^l(\tau_{rzj}^t, \tau_{rx}^s, \tau_{rz}^i)$ is the stress component corresponding to $w_j^l(w_j^t\, w^s, w^i)$. The condition of continuity of displacement at P_j^n (a_n, θ_j, z) on C_j^n gives

$$\sum_{m=-\infty}^{\infty} [B_{jm}^n H_m(k_{sh}^n a_n) \exp(im\theta_j) + C_{jm}^n J_m(k_{sh}^n a_n) \exp(im\theta_j)]$$
$$= \left[w_0 \exp(ik_{sh}x) + \sum_{i=1}^{N} \sum_{m=-\infty}^{\infty} A_{im} H_m(k_{sh} r_i) \exp(im\theta_i) \right]_{P_j^n}. \tag{14}$$

Multiplying by $\exp[-iv\theta_j]$ and integrating from 0 to 2π, we have

$$B_{jv}^n H_v(k_{sh}^n a_n) + C_{jv}^n J_v(k_{sh}^n a_n) = w_0 i^v J_v(k_{sh} a_n) \exp(ik_{sh} r_{jo} \cos\theta_{jo}) + \sum_{i=1}^{N} \sum_{m=-\infty}^{\infty} A_{im} K_{ijmv}, \tag{15}$$

where

$$K_{ijmv} = \frac{1}{2\pi} \int_0^{2\pi} [H_m(k_{sh} r_i) \exp(im\theta_i)]_{P_j^n} \exp(-iv\theta_j)\, d\theta_j \quad (i \neq j)$$
$$= H_v(k_{sh} a_n) \delta_{mv} \quad (i = j), \tag{16}$$

δ_{mv} is the Kronecker delta. Using the addition theorem of Hankel functions (Bose and Mal, 1973), we get

$$K_{ijmv} = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \exp(im\theta_{ji})(-1)^m \sum_{s=-\infty}^{\infty} (-1)^s J_s(k_{sh} a_n) \right.$$
$$\left. \times H_{s-m}(k_{sh} r_{ji}) \exp[is(\theta_j - \theta_{ji})] \right\} \exp(-iv\theta_j)\, d\theta_j$$
$$= J_v(k_{sh} a_n) H_{m-v}(k_{sh} r_{ji}) \exp[i(m-v)\theta_{ji}] \quad (i \neq j), \tag{17}$$

where (r_{ji}, θ_{ji}) are the polar coordinates of o_j referred to o_i as origin. Thus, eqn (15) becomes

$$B^n_{j\nu}H_\nu(k^n_{\rm sh}a_n)+C^n_{j\nu}J_\nu(k^n_{\rm sh}a_n) = A_{j\nu}H_\nu(k_{\rm sh}a_n)$$
$$+J_\nu(k_{\rm sh}a_n)\left[w_0 i^\nu \exp(ik_{\rm sh}r_{jo}\cos\theta_{jo})+\sum_{i=1}^{N}{}' \sum_{m=-\infty}^{\infty} A_{i,m+\nu}H_m(k_{\rm sh}r_{ji})\exp(im\theta_{ji})\right], \tag{18}$$

where Σ' denotes the sum over all cylindrical inclusions except the jth. The conditions of continuity of displacement at P^l_j (a_l, θ_j, z) $(l = 1 \sim n-1)$ on C^l_j and P^0_j (a_0, θ_j, z) on C^0_j give

$$B^l_{j\nu}H_\nu(k^l_{\rm sh}a_l)+C^l_{j\nu}J_\nu(k^l_{\rm sh}a_l) = B^{l+1}_{j\nu}H_\nu(k^{l+1}_{\rm sh}a_l)+C^{l+1}_{j\nu}J_\nu(k^{l+1}_{\rm sh}a_l) \tag{19}$$

$$D_{j\nu}J_\nu(k^0_{\rm sh}a_0) = B^1_{j\nu}H_\nu(k^1_{\rm sh}a_0)+C^1_{j\nu}J_\nu(k^1_{\rm sh}a_0). \tag{20}$$

The conditions of continuity of shear stress at P^n_j, P^l_j $(l = 1 \sim n-1)$ and P^0_j similarly give

$$\frac{\mu_n}{\mu}\left[B^n_{j\nu}\frac{\partial}{\partial a_n}H_\nu(k^n_{\rm sh}a_n)+C^n_{j\nu}\frac{\partial}{\partial a_n}J_\nu(k^n_{\rm sh}a_n)\right] = A_{j\nu}\frac{\partial}{\partial a_n}H_\nu(k_{\rm sh}a_n)$$
$$+\frac{\partial}{\partial a_n}J_\nu(k_{\rm sh}a_n)\left[w_0 i^\nu \exp(ik_{\rm sh}r_{jo}\cos\theta_{jo})+\sum_{i=1}^{N}{}' \sum_{m=-\infty}^{\infty} A_{i,m+\nu}H_m(k_{\rm sh}r_{ji})\exp(im\theta_{ji})\right] \tag{21}$$

$$\frac{\mu_l}{\mu_{l+1}}\left[B^l_{j\nu}\frac{\partial}{\partial a_l}H_\nu(k^l_{\rm sh}a_l)+C^l_{j\nu}\frac{\partial}{\partial a_l}J_\nu(k^l_{\rm sh}a_l)\right] = B^{l+1}_{j\nu}\frac{\partial}{\partial a_l}H_\nu(k^{l+1}_{\rm sh}a_l)+C^{l+1}_{j\nu}\frac{\partial}{\partial a_l}J_\nu(k^{l+1}_{\rm sh}a_l) \tag{22}$$

$$\frac{\mu_0}{\mu_1}D_{j\nu}\frac{\partial}{\partial a_0}J_\nu(k^0_{\rm sh}a_0) = B^1_{j\nu}\frac{\partial}{\partial a_0}H_\nu(k^1_{\rm sh}a_0)+C^1_{j\nu}\frac{\partial}{\partial a_0}J_\nu(k^1_{\rm sh}a_0). \tag{23}$$

From eqns (18)–(23), the unknown $A_{j\nu}$ is found to be

$$A_{j\nu} = A'_\nu F_{j\nu}, \tag{24}$$

where

$$A'_\nu = X_\nu\frac{H_\nu(k^n_{\rm sh}a_n)}{H_\nu(k_{\rm sh}a_n)}+Y_\nu\frac{J_\nu(k^n_{\rm sh}a_n)}{H_\nu(k_{\rm sh}a_n)}-\frac{J_\nu(k_{\rm sh}a_n)}{H_\nu(k_{\rm sh}a_n)} \tag{25}$$

$$F_{j\nu} = w_0 i^\nu \exp(ik_{\rm sh}r_{jo}\cos\theta_{jo})+\sum_{i=1}^{N}{}' \sum_{m=-\infty}^{\infty} A'_{m+\nu}F_{i,m+\nu}H_m(k_{\rm sh}r_{ji})\exp(im\theta_{ji}). \tag{26}$$

In eqn (25), X_ν and Y_ν are given in the Appendix.

3. THE AVERAGE FIELD FOR A RANDOM DISTRIBUTION OF CYLINDRICAL INCLUSIONS

We consider the positions of the cylindrical inclusions to be random. If we denote the position vector of o_i by $\mathbf{r}_{io}$ and the probability density of the random variable $(\mathbf{r}_{1o}, \mathbf{r}_{2o}, \ldots, \mathbf{r}_{No})$ by $p(\mathbf{r}_{1o}, \mathbf{r}_{2o}, \ldots, \mathbf{r}_{No})$, then due to the indistinguishability of the cylindrical inclusions, it is symmetric in its arguments and we have (Waterman and Truell, 1961)

$$p(\mathbf{r}_{1o}, \mathbf{r}_{2o}, \ldots, \mathbf{r}_{No}) = p(\mathbf{r}_{io})\mathbf{p}(\mathbf{r}_{1o}, \mathbf{r}_{2o}, \ldots, ', \ldots, \mathbf{r}_{No}|\mathbf{r}_{io})$$
$$= p(\mathbf{r}_{io})p(\mathbf{r}_{jo}|\mathbf{r}_{io})p(\mathbf{r}_{1o}, \mathbf{r}_{2o}, \ldots, ', \ldots, ', \ldots, \mathbf{r}_{No}|\mathbf{r}_{jo}, \mathbf{r}_{io})$$
$$p(\mathbf{r}_{io}) = p(\mathbf{r}_{1o}), \quad p(\mathbf{r}_{jo}|\mathbf{r}_{io}) = p(\mathbf{r}_{2o}|\mathbf{r}_{1o}) \quad (i \neq j), \tag{27}$$

where the vertical lines in the arguments denote the usual conditional probabilities. A prime in the first part of eqn (27) means $\mathbf{r}_{io}$ is absent, while two primes in the second part of eqn (27) mean both $\mathbf{r}_{io}$ and $\mathbf{r}_{jo}$ are absent. For a uniform composite, the positions of a single cylindrical inclusion are equally probable within a large region S of a cross-section of the material and, hence, its distribution is uniform with density

$$\begin{aligned} p(\mathbf{r}_{io}) &= \frac{1}{S} \quad \mathbf{r}_{io} \in S \\ &= 0 \quad r_{io} \notin S. \end{aligned} \tag{28}$$

If now o_i, well within S, is held fixed, the distribution of the cylindrical inclusions around the cylindrical inclusion will be circularly symmetrical. Thus, $p(\mathbf{r}_{jo}|\mathbf{r}_{io})$ is a function of r_{ij} alone, and we can write

$$\begin{aligned} p(\mathbf{r}_{jo}|\mathbf{r}_{io}) &= \frac{1}{S}[1-g(r_{ij})] \quad \mathbf{r}_{jo} \in S \\ &= 0 \quad \mathbf{r}_{jo} \notin S, \end{aligned} \tag{29}$$

where the pair correlation function $g(r_{ij}) \leqslant 1$ is a decreasing function of r_{ij}. The normalization condition gives, in the limit as $S \to \infty$

$$\lim_{R\to\infty} \frac{1}{R^2} \int_0^R g(r_{ij}) r_{ij}\, \mathrm{d}r_{ij} = 0. \tag{30}$$

Due to the impossibility of interpenetration of the cylindrical inclusions and their independence when they are infinitely apart, we have

$$\begin{aligned} g(r_{ij}) &= 1 \quad r_{ij} < 2a_n \\ &= 0 \quad r_{ij} \to \infty. \end{aligned} \tag{31}$$

A function satisfying these conditions is

$$\begin{aligned} g(r_{ij}) &= 1 \quad r_{ij} < 2a_n \\ &= V\exp(-r_{ij}/L) \quad r_{ij} \geqslant 2a_n \end{aligned} \tag{32}$$

where V ($0 < V \leqslant \exp[2a_n/L]$) is the coefficient and $L > 0$ is the correlation length.

We denote the conditional expectations of a statistical quantity f when either o_i or o_i and o_j together are held fixed as

$$\begin{aligned} \langle f \rangle_i &= \int \cdots' \cdots \int f p(\mathbf{r}_{1o}, \ldots' \ldots, \mathbf{r}_{No}|\mathbf{r}_{io})\, \mathrm{d}\tau_1 \ldots' \ldots \mathrm{d}\tau_N \\ \langle f \rangle_{ij} &= \int \cdots'' \cdots \int f p(\mathbf{r}_{1o}, \ldots'' \ldots, \mathbf{r}_{No}|\mathbf{r}_{io}, \mathbf{r}_{jo})\, \mathrm{d}\tau_1 \ldots'' \ldots \mathrm{d}\tau_N, \end{aligned} \tag{33}$$

where $\mathrm{d}\tau_i$ ($i = 1 \sim N$) is the volume element at $\mathbf{r}_{io}$. To determine $\langle F_{iv} \rangle_i$ of eqn (26) we take the conditional expectation to obtain

$$\langle F_{iv}\rangle_i = w_0 i^v \exp(ik_{sh}r_{io}\cos\theta_{io}) + n_0\left(1-\frac{1}{N}\right)\sum_{m=-\infty}^{\infty} A'_{m+v} \times \int_{\mathbf{r}_{io},\mathbf{r}_{jo}\in S} [1-g(r_{ij})]\langle F_{j,m+v}\rangle_{ij} H_m(k_{sh}r_{ij})\exp(im\theta_{ji})\,\mathrm{d}\tau_j, \quad (34)$$

where $n_0 = N/S = c/\pi a_0^2$ is the number of cylindrical inclusions per unit area and c is the volume concentration of inclusions in the matrix. Equation (34) involves the conditional expectation with two cylindrical inclusions held fixed. If we take the conditional expectation of eqn (26) with two cylindrical inclusions held fixed, the resulting equation will contain the conditional expectation with three cylindrical inclusions held fixed, and so on. We shall eliminate this hierarchy by assuming Lax's quasicrystalline approximation (Lax, 1952), which involves the two-inclusion correlation function and implies

$$\langle F_{iv}\rangle_{ij} = \langle F_{iv}\rangle_i, \quad i \neq j. \quad (35)$$

According to the extinction theorem when S and N become infinitely large (Lax, 1952), the incident wave is extinguished on entering the composite, so that the corresponding term in eqn (34) can be dropped. Thus, this equation reduces to

$$\langle F_{im}\rangle_i = n_0 \sum_{v=-\infty}^{\infty} A'_{m+v} \int_{|\mathbf{r}_{jo}-\mathbf{r}_{io}|>2a_n} [1-g(r_{ji})]\langle F_{j,m+v}\rangle_j H_v(k_{sh}r_{ji})\exp(iv\theta_{ji})\,\mathrm{d}\tau_j. \quad (36)$$

Assuming the existence of an average plane wave, we try, for eqn (36), the solution

$$\langle F_{im}\rangle_i = i^m F_m \exp(iK_{sh}x_{io}), \quad x_{io} = r_{io}\cos\theta_{io}, \quad (37)$$

where F_m is a constant and K_{sh} is the wave number of the effective SH wave. Making use of the Green's theorem and the plane wave expansion

$$\exp(iK_{sh}x_{jo}) = \exp(iK_{sh}x_{io}) \sum_{s=-\infty}^{\infty} i^{-s} J_s(K_{sh}r_{ji})\exp(-is\theta_{ji}), \quad (38)$$

we find that the first integral appearing in eqn (36) becomes

$$\begin{aligned}\int_{|\mathbf{r}_{jo}-\mathbf{r}_{io}|>2a_n} & \exp(iK_{sh}x_{jo})H_v(k_{sh}r_{ji})\exp(iv\theta_{ji})\,\mathrm{d}\tau_j \\ &= \frac{1}{k_{sh}^2-K_{sh}^2}\int_{|\mathbf{r}_{jo}-\mathbf{r}_{io}|>2a_n} \{\nabla^2[\exp(iK_{sh}x_{jo})]H_v(k_{sh}r_{ji})\exp(iv\theta_{ji}) \\ &\quad - \exp(iK_{sh}x_{jo})\nabla^2[H_v(k_{sh}r_{ji})\exp(iv\theta_{ji})]\}\,\mathrm{d}\tau_j \\ &= \exp(iK_{sh}x_{io})\frac{2\pi a_n i^{-v}}{k_{sh}^2-K_{sh}^2}\left[J_v(2K_{sh}a_n)\frac{\partial}{\partial a_n}H_v(2k_{sh}a_n) - H_v(2k_{sh}a_n)\frac{\partial}{\partial a_n}J_v(2K_{sh}a_n)\right]. \end{aligned} \quad (39)$$

The second integral in eqn (36) can be also simplified by using the above expansion and eqn (36) reduces to the system of equations

$$\begin{aligned} F_m = 2\pi n_0 \sum_{v=-\infty}^{\infty} A'_{m+v}F_{m+v} & \times \left\{\frac{a_n}{k_{sh}^2-K_{sh}^2}\left[J_v(2K_{sh}a_n)\frac{\partial}{\partial a_n}H_v(2k_{sh}a_n) - H_v(2k_{sh}a_n)\frac{\partial}{\partial a_n}J_v(2K_{sh}a_n)\right]\right. \\ & \left. - \int_{2a_n}^{\infty} g(r_{ji})J_v(K_{sh}r_{ji})H_v(k_{sh}r_{ji})r_{ji}\,\mathrm{d}r_{ji}\right\}. \end{aligned} \quad (40)$$

The elimination of F_m from the above equations yields a determinantal equation of infinite order for K_{sh}. The effects of multiple scattering on the coherent wave are of great practical importance for the concentration $c = 0.01 \sim 0.4$. At very low concentrations ($c < 0.01$) multiple scattering can be neglected and each scatterer can be treated as independent.

Assuming $k_{sh}a_0$ and L to be sufficiently small compared to the wavelength, we obtain by expanding the Bessel and Hankel functions and retaining the lowest order terms for $h/a_0 = 0.0$

$$F_m \simeq -\pi c \sum_{v=-\infty}^{\infty} A''_{m+v} F_{m+v} \left[\frac{1}{\pi} \left(\frac{K_{sh}}{k_{sh}} \right)^{v} \frac{1}{1 - \left(\frac{K_{sh}}{k_{sh}} \right)^2} + \frac{i}{2} k_{sh}^2 I_v \right] \tag{41}$$

where

$$A''_0 = \frac{\rho_0}{\rho} - 1$$

$$A''_{\pm 1} = \frac{\mu - \mu_0}{\mu_0 + \mu}$$

$$A''_m = 0, \quad (|m| \geqslant 2) \tag{42}$$

$$I_0 \simeq \frac{2i}{\pi} VL^2 \left(1 + \log \frac{k_{sh}L}{2} - \frac{i\pi}{2} \right)$$

$$I_1 \simeq -\frac{i}{\pi} VL^2 \frac{K_{sh}}{k_{sh}}$$

$$I_2 \simeq -\frac{i}{2\pi} VL^2 \left(\frac{K_{sh}}{k_{sh}} \right)^2$$

$$I_m \simeq 0, \quad m \geqslant 3. \tag{43}$$

The effective shear modulus μ^*_{xz} can be easily obtained from the phase velocity $\mathrm{Re}(k_{sh}/K_{sh})$ of the effective SH wave as follows:

$$\mu^*_{xz} = \mu \left(\frac{\rho^*}{\rho} \right) \left[\mathrm{Re} \left(\frac{k_{sh}}{K_{sh}} \right) \right]^2, \tag{44}$$

where the average mass density ρ^* is

$$\rho^* = \rho \left[1 - c \left(1 + \frac{h}{a_0} \right)^2 \right] + \rho_0 c + \sum_{l=1}^{n} \rho_l c \left[\frac{1}{n} \frac{h}{a_0} \left(2 + \frac{2l-1}{n} \frac{h}{a_0} \right) \right]. \tag{45}$$

4. NUMERICAL RESULTS AND DISCUSSIONS

To examine the effect of interface properties on the phase velocity and attenuation of coherent plane wave through the composite medium, for a given value of $k_{sh}a_0$, A'_v is computed. Next, the complex coefficient matrix M corresponding to F_m [eqn (40)] is formed. The complex determinant of the coefficient matrix is computed using standard Gauss elimination techniques. For a given $k_{sh}a_0$, the root of the equation $\det M = 0$ is searched in the complex K_{sh} plane using Muller's method. Good initial guesses are provided by eqn

Table 1. Material properties of SiC and Al

	ρ_0 (kg m^{-3})	μ_0 (GPa)	v_0		ρ (kg m^{-3})	μ (GPa)	v
SiC	3181	188.1	0.17	Al	2706	26.7	0.34

(41) at low values of $k_{sh}a_0$ and these can be used systematically to obtain quick convergence of roots at increasingly higher values of $k_{sh}a_0$. The considered composite was an SiC–Al composite. The constituent properties are given in Table 1. Three special cases of interface material are considered. The elastic properties of Case I, II and III are given by

Case I

$$\mu_{\rm I}(r) = \frac{\mu+\mu_0}{2}$$

$$\rho_{\rm I} = \frac{\rho+\rho_0}{2} \quad (a_0 \leqslant r \leqslant a_0+h). \tag{46}$$

Case II

$$\mu_{\rm II}(r) = (\mu-\mu_0)\left(\frac{r-a_0}{h}\right)+\mu_0$$

$$\rho_{\rm II}(r) = (\rho-\rho_0)\left(\frac{r-a_0}{h}\right)+\rho_0 \quad (a_0 \leqslant r \leqslant a_0+h). \tag{47}$$

Case III

$$\mu_{\rm III}(r) = 4(\mu-\mu_0)\left[\frac{r-(a_0+h/2)}{h}\right]^3+\frac{\mu+\mu_0}{2}$$

$$\rho_{\rm III}(r) = 4(\rho-\rho_0)\left[\frac{r-(a_0+h/2)}{h}\right]^3+\frac{\rho+\rho_0}{2} \quad (a_0 \leqslant r \leqslant a_0+h). \tag{48}$$

The material properties of the layers given above are calculated at the midpoint of each layer assuming variations of Cases I, II and III from the boundary of the inclusion to the matrix medium.

Figure 2 shows the variation of the phase velocity $\mathrm{Re}(k_{sh}/K_{sh})$ of the effective SH wave with the number of layers n for Case II and $c = 0.3$, $h/a_0 = 0.1$, $a_0\omega/c_{sh} = 1.0$. Case II refers to the case of the interface material through which the elastic properties vary linearly from those of the inclusions to those of the matrix. It is found that the truncation after $n = 30$ gives practically adequate results for Case II. The effect of the interface layer on $\mathrm{Re}(k_{sh}/K_{sh})$ at $a_0\omega/c_{sh} = 1.0$ for $c = 0.3$ is shown in Fig. 3. The figure shows that the phase velocity $\mathrm{Re}(k_{sh}/K_{sh})$ increases with the h/a_0 ratio, and depends on the constituents and the nature of the interface layer. The phase velocity for Cases II and III, which was evaluated by taking $n = 30$ and 32, agreed to at least three decimal places. Thus, it may be said that the result for $n = 30$ is, from a practical view point, quite satisfactory.

In Fig. 4, the phase velocity $\mathrm{Re}(k_{sh}/K_{sh})$ of the effective SH wave is plotted as function of the frequency $a_0\omega/c_{sh}$ for $c = 0.3$. The dashed curve refers to the case $h/a_0 = 0.0$ and the solid curve refers to $h/a_0 = 0.1$. The interface material for Case III is considered. The phase velocity increases with the frequency, reaches a maximum, and then decreases, and the interface effect increases the phase velocity. Figure 5 shows the variation of the attenuation $\mathrm{Im}(k_{sh}/k_{sh})$ of the effective SH wave with the frequency $a_0\omega/c_{sh}$ for Case III and $c = 0.3$, $h/a_0 = 0.0, 0.1$. The attenuation decreases with the frequency, reaches a minimum, and then increases, and the interface effect increases the attenuation. The computations carried out

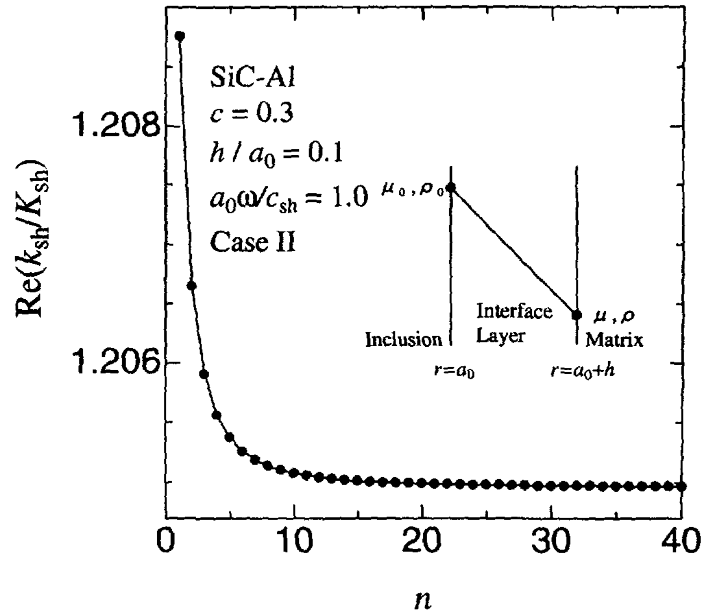

Fig. 2. Phase velocity vs n.

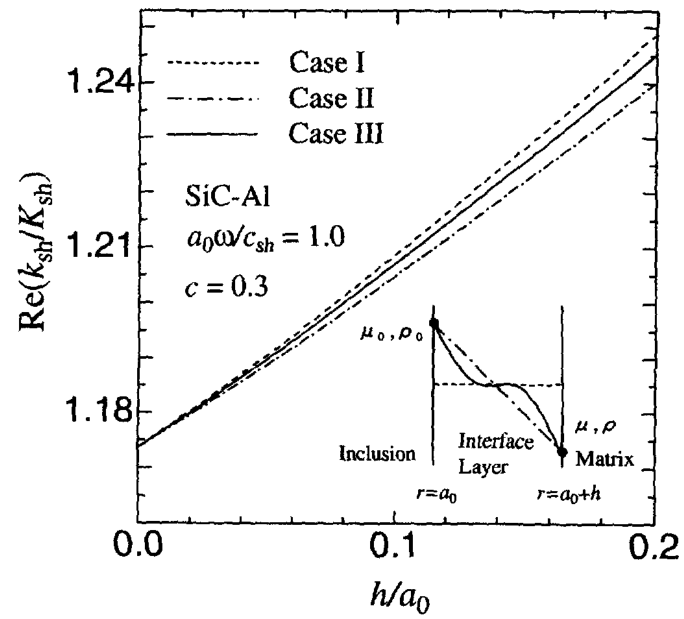

Fig. 3. Phase velocity vs h/a_0.

reveal that the truncation after $n = 30$ gives practically adequate results at any desired finite frequency for Cases II and III.

Figure 6 shows the variation of the effective shear modulus μ_{xz}^* of SiC–Al with the frequency $a_0\omega/c_{sh}$ for Case III and $c = 0.3$, $h/a_0 = 0.0$, 0.1. The effective shear modulus increases with the frequency, reaches a maximum, and then decreases, and the interface effect increases the effective shear modulus. As $a_0\omega/c_{sh} \to 0$, the dynamic effective shear modulus tends to the static solution. Using the Eshelby method, we obtain the effective shear modulus μ_{xz}^* for $h/a_0 = 0.0$ as Wakashima (1976)

$$\mu_{xz}^* = (1-c)\mu + c\mu_0 - c(1-c)\frac{(\mu_0-\mu)^2}{(1-c)\mu_0+(1+c)\mu}. \tag{49}$$

Making use of the law of mixture, we also have

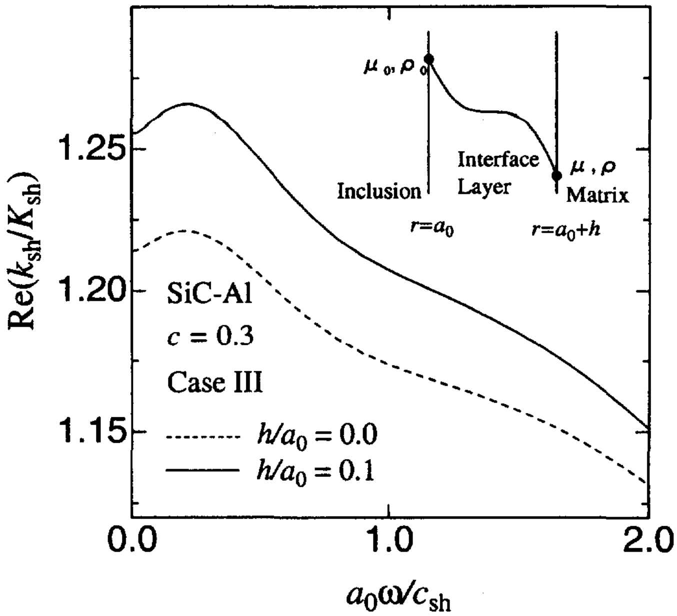

Fig. 4. Effect of interface on phase velocity vs frequency.

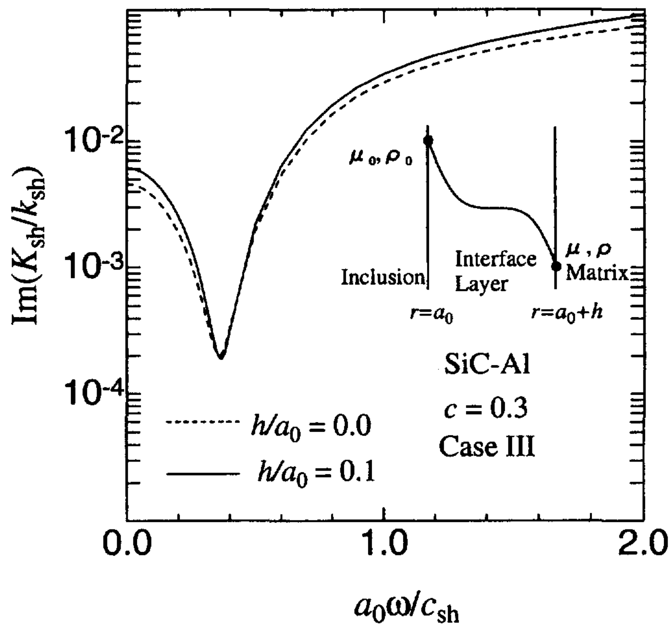

Fig. 5. Effect of interface on attenuation vs frequency.

$$\mu_{xz}^* = \frac{\mu\mu_0}{(1-c)\mu_0 + c\mu}. \tag{50}$$

Figure 7 shows the variation of the static effective shear modulus μ_{xz}^* with the volume concentration c for $h/a_0 = 0.0$. A comparison of the static effective shear modulus is made in $a_0\omega/c_{sh} = 0$, Eshelby method and law of mixture. The results agree much better with those obtained from the Eshelby method in the lower concentration. As mentioned earlier, the Lax's quasicrystalline approximation is valid for the small concentration ($c < 0.4$) or to second-order in the concentration c.

In conclusion, the multiple scattering of antiplane shear waves by cylindrical inclusions with thick nonhomogeneous interface layers was analyzed. The interface effect can increase phase velocity, attenuation of coherent plane wave in a metal matrix composite and effective

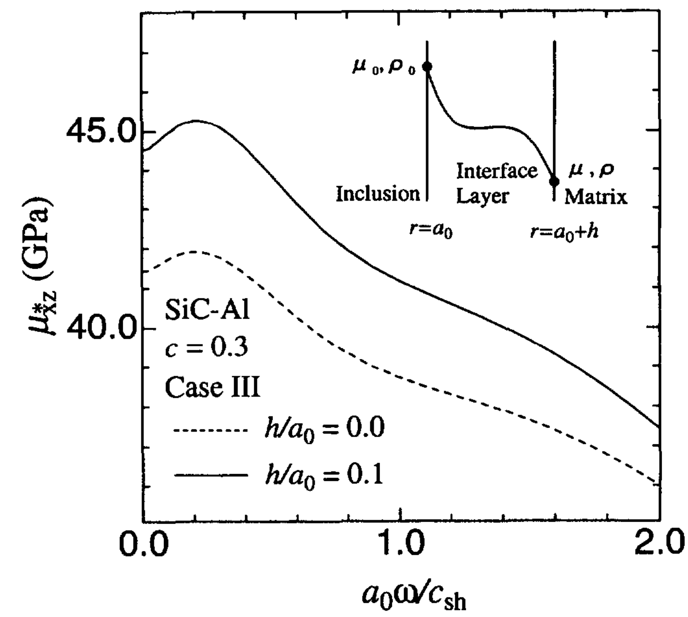

Fig. 6. Effect of interface on effective shear modulus vs frequency.

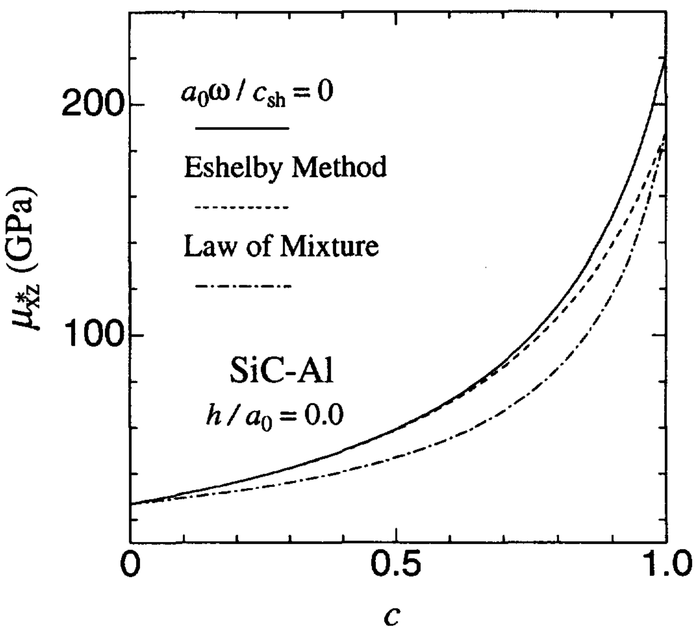

Fig. 7. Static effective shear modulus vs concentration.

elastic constant, and depends on the frequency and the material properties of the interface layers. The numerical results at the volume concentration of inclusions $c = 0.3$ were obtained for any given finite frequency, and layers with nonhomogeneous elastic properties of any desired finite thickness.

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APPENDIX

X_ν, Y_ν in eqn (25) are

$$X_\nu = \frac{G_\nu Q_\nu^n}{E_\nu Q_\nu^n - U_\nu P_\nu^n}$$

$$Y_\nu = \frac{-G_\nu P_\nu^n}{E_\nu Q_\nu^n - U_\nu P_\nu^n}, \tag{A1}$$

where

$$E_\nu = H_\nu(k_{sh}^n a_n)\frac{\partial}{\partial a_n}H_\nu(k_{sh}a_n) - H_\nu(k_{sh}a_n)\frac{\partial}{\partial a_n}H_\nu(k_{sh}^n a_n)\frac{\mu_n}{\mu}$$

$$U_\nu = J_\nu(k_{sh}^n a_n)\frac{\partial}{\partial a_n}H_\nu(k_{sh}a_n) - H_\nu(k_{sh}a_n)\frac{\partial}{\partial a_n}J_\nu(k_{sh}^n a_n)\frac{\mu_n}{\mu}$$

$$G_\nu = J_\nu(k_{sh}a_n)\frac{\partial}{\partial a_n}H_\nu(k_{sh}a_n) - H_\nu(k_{sh}a_n)\frac{\partial}{\partial a_n}J_\nu(k_{sh}a_n). \tag{A2}$$

The recurrence formula for P_ν^n, Q_ν^n are given by

$$P_\nu^{l+1} = \frac{P_\nu^l}{Q_\nu^l}K_\nu^l + M_\nu^l$$

$$Q_\nu^{l+1} = \frac{P_\nu^l}{Q_\nu^l}L_\nu^l + N_\nu^l \quad (l = 1 \sim n-1) \tag{A3}$$

$$P_\nu^1 = H_\nu(k_{sh}^1 a_0)\frac{\mu_0}{\mu_1}\frac{\partial}{\partial a_0}J_\nu(k_{sh}^0 a_0) - J_\nu(k_{sh}^0 a_0)\frac{\partial}{\partial a_0}H_\nu(k_{sh}^1 a_0)$$

$$Q_\nu^1 = J_\nu(k_{sh}^1 a_0)\frac{\mu_0}{\mu_1}\frac{\partial}{\partial a_0}J_\nu(k_{sh}^0 a_0) - J_\nu(k_{sh}^0 a_0)\frac{\partial}{\partial a_0}J_\nu(k_{sh}^1 a_0). \tag{A4}$$

In eqn (A3), $K_\nu^l, L_\nu^l, M_\nu^l, N_\nu^l$ $(l = 1 \sim n-1)$ are

$$K_\nu^l = \left[J_\nu(k_{sh}^l a_l)\frac{\partial}{\partial a_l}H_\nu(k_{sh}^{l+1}a_l) - H_\nu(k_{sh}^{l+1}a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}J_\nu(k_{sh}^l a_l)\right]$$
$$\Bigg/\left[J_\nu(k_{sh}^l a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}H_\nu(k_{sh}^l a_l) - H_\nu(k_{sh}^l a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}J_\nu(k_{sh}^l a_l)\right]$$

$$L_\nu^l = \left[J_\nu(k_{sh}^l a_l)\frac{\partial}{\partial a_l}J_\nu(k_{sh}^{l+1}a_l) - J_\nu(k_{sh}^{l+1}a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}J_\nu(k_{sh}^l a_l)\right]$$
$$\Bigg/\left[J_\nu(k_{sh}^l a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}H_\nu(k_{sh}^l a_l) - H_\nu(k_{sh}^l a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}J_\nu(k_{sh}^l a_l)\right]$$

$$M_\nu^l = \left[H_\nu(k_{sh}^l a_l)\frac{\partial}{\partial a_l}H_\nu(k_{sh}^{l+1}a_l) - H_\nu(k_{sh}^{l+1}a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}H_\nu(k_{sh}^l a_l)\right]$$
$$\Bigg/\left[H_\nu(k_{sh}^l a_l)\frac{\mu_l}{\mu_l+1}\frac{\partial}{\partial a_l}J_\nu(k_{sh}^l a_l) - J_\nu(k_{sh}^l a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}H_\nu(k_{sh}^l a_l)\right]$$

$$N_\nu^l = \left[H_\nu(k_{sh}^l a_l)\frac{\partial}{\partial a_l}J_\nu(k_{sh}^{l+1}a_l) - J_\nu(k_{sh}^{l+1}a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}H_\nu(k_{sh}^l a_l)\right]$$
$$\Bigg/\left[H_\nu(k_{sh}^l a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}J_\nu(k_{sh}^l a_l) - J_\nu(k_{sh}^l a_l)\frac{\mu_l}{\mu_{l+1}}\frac{\partial}{\partial a_l}H_\nu(k_{sh}^l a_l)\right] \tag{A5}$$